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SHELL MODEL DESCRIPTION OF INTERACTING BOSONS

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The interacting boson model, describing collective states of even–even nuclei, is introduced as a drastic truncation of large scale shell model calculations. The shell model hamiltonian can be diagonalized by using a correspondence, or mapping, of the nucleon states in the truncated space into states obtained by coupling proton and neutron s- and d-bosons. The equivalent boson hamiltonian in a simple case is obtained and diagonalized. Eigenstates with definite proton–neutron symmetry (good F-spin) emerge for certain values of proton and neutron numbers. In general the situation is more complex but the results obtained follow closely the experimental data.

In a previous note [1] we suggested a correspondence between pair states of identical nucleons with $J = 0$ and $J = 2$ and states with s- and d-bosons. We then considered couplings of proton and neutron bosons. In case where the boson hamiltonian had a certain symmetry, resulting states were characterized by their symmetry properties or by their F-spin eigenvalues. States with maximum symmetry ($F = N/2$) were identified with states of the interacting bosons introduced by Arima and Iachello [2].

In the present paper we discuss how to construct a hamiltonian (and other relevant operators) of proton and neutron bosons starting from a microscopic shell-model description. Our procedure consists in first truncating the very large shell model space required for heavier nuclei and then replacing calculations in this truncated space by much simpler calculations using boson states and operators. We show that the various situations observed in heavy even–even nuclei (described sometimes as anharmonic vibrations, axial and triaxial rotations, etc.) all emerge naturally in our scheme. Finally, we study in which cases F-spin is expected to be a good quantum number for nuclear states.

An example will demonstrate what drastic truncation must be carried out on shell model calculations to make them possible. In ^{154}Sm the 12 valence protons can occupy all orbits in the 50–82 shell and the 10 valence neutrons those between 82 and 126. Even if we ignore excitations into higher shells or core excitations, the number of shell model states is overwhelming. There are 41, 654, 193, 516, 797 positive parity states with $J = 0$; 346, 132, 052, 934, 889 states with $J = 2$ and 530, 897, 397, 260, 575 states with $J = 4$! Since all orbits in the 50–82 and 82–126 shells are close to each other, a truncation based on omission or orbits is not acceptable. However, we are not interested in most of these billions of states. Apart from a small number, they cannot be identified in an actual nucleus. We are really interested only in a small, se-

lected set of states, like the low-lying collective states of even nuclei. The truncation procedure, or coupling scheme, which we are looking for should reproduce, to a good approximation, properties of these states. It should be, at the same time, simple enough to enable sensible calculations as well as to offer physical insight into the nature of the states considered.

In constructing the truncated space, we make use of the correspondence between fermions and boson states discussed in ref. [1], and introduce the pair creation operators [3]:

$$S_\pi^+ = \sum_{j_\pi} \alpha_{j_\pi} S_{j_\pi}^+, \qquad S_\nu^+ = \sum_{j_\nu} \alpha_{j_\nu} S_{j_\nu}^+,$$

$$D_{\pi,\mu}^+ = \sum_{j_\pi, j_\pi'} \beta_{j_\pi j_\pi'} D_{j_\pi j_\pi', \mu}^+ \tag{1}$$

$$D_{\nu,\mu}^+ = \sum_{j_\nu, j_\nu'} \beta_{j_\nu j_\nu'} D_{j_\nu j_\nu', \mu}^+,$$

where the subscript $\pi(\nu)$ refers to protons (neutrons). The truncated subspace, which we call S–D, contains all states of the form

$$[\{(S_\pi^+)^{n_{s_\pi}} (D_\pi^+)^{n_{d_\pi}}_{\gamma_\pi J_\pi}\} \times \{(S_\nu^+)^{n_{s_\nu}} (D_\nu^+)^{n_{d_\nu}}_{\gamma_\nu J_\nu}\}]_{JM} |0\rangle, \tag{2}$$

with

$$n_{s_\pi} + n_{d_\pi} = \tfrac{1}{2} n_\pi \equiv N_\pi, \qquad n_{s_\nu} + n_{d_\nu} = \tfrac{1}{2} n_\nu \equiv N_\nu, \tag{3}$$

where n_π (n_ν) is the number of valence protons (neutrons).

As we have pointed out in ref. [1], the states $(S^+)^{n_s}(D^+)^{n_d}_{\gamma JM}|0\rangle$ are not orthogonal and to make them so we have to project out all components of the form $(S^+)^{n_s+1} B^+_{JM}|0\rangle$, where B^+_{JM} are all operators which create states with $2(n_d - 1)$ nucleons. The proton (neutron) states which appear in eq. (2) are the states obtained after this projection. Such states with different values of n_d are then orthogonal to each other. For each value of n_d (and n_s), the quantum numbers γ_π (γ_ν) characterize an orthogonal set of states obtained from the independent modes of coupling which yield the same value of J_π (J_ν). We note that when there are in the major shell more protons than proton holes, or more neutrons than neutron holes, the corresponding pair operators are constructed from proton (neutron) holes and n_π (n_ν) in eq. (3) stands for the number of proton (neutron) holes.

In the first step we calculate the matrix elements of the shell model hamiltonian:

$$H = H_\pi + H_\nu + H_{\pi\nu}, \tag{4}$$

in the S–D subspace defined by eq. (2). The reasons why the restriction to the S–D subspace may be a good approximation are related to the main features of the shell-model hamiltonian (4). When considered separately, H_π and H_ν have nuclear eigenstates with good seniority or generalized seniority as indicated experimentally by the behavior of the separation energies and by the constant spacing between $J = 0$, $J = 2$ states in semi-magic nuclei [3]. On the other hand, the strong and attractive proton–neutron interaction $H_{\pi\nu}$, in cases where it was possible to determine it from experiment, is dominated by the seniority breaking quadrupole–quadrupole term $H_{\pi\nu} = -kT_\pi^{(2)} \cdot T_\nu^{(2)}$. This interaction has large matrix elements between states in the S–D subspace while having small matrix elements leading out of this subspace. For example, the matrix element between the states $[(S_\pi^+)^{N_\pi - 1} D_\pi^+ (S_\nu^+)^{N_\nu}]_{J=2}|0\rangle$ and $[(S_\pi^+)^{N_\pi}(S_\nu^+)^{N_\nu - 1} D_\nu^+]_{J=2}|0\rangle$ is large, causing a lowering of the first excited $J = 2$ state in nuclei with both active protons and neutrons. Since the calculation of the matrix elements of H is in general very complicated it must be carried out numerically. We have explicitly verified in several simple shell model calculations [4] that the biggest components of the wave functions for the states we want to describe are those within the S–D subspace. States outside this space have small amplitudes and their contribution to transitions and energies can probably be included by introducing a renormalized hamiltonian which takes into account the effects of the truncation.

In the next step we map fermion operators defined in the S–D subspace of eq. (2) onto boson operators defined in a space, called s–d, generated by the boson operators $s_\pi^+, d_\pi^+, s_\nu^+, d_\nu^+$:

$$[\{(s_\pi^+)^{n_{s_\pi}} (d_\pi^+)^{n_{d_\pi}}_{\gamma_\pi J_\pi}\} \times \{(s_\nu^+)^{n_{s_\nu}} (d_\nu^+)^{n_{d_\nu}}_{\gamma_\nu J_\nu}\}]\ |0\rangle. \tag{5}$$

Among the states (5), those with different values of n_{d_π} (or n_{d_ν}) (different numbers of proton (or neutron) d-bosons) are orthogonal. These correspond to states (2) constructed by applying the same number of D_π^+

(or D_ν^+) operators. Any state not allowed by the coupling of n_{d_π} (or n_{d_ν}) d-bosons, because of symmetry, cannot be obtained by coupling the same number of D^+ fermion pair operators. On the other hand, there are allowed boson states which cannot be obtained from nucleon pairs due to the Pauli principle. As the number of D_π^+ (or D_ν^+) operators is increased such effects may become important, and a detailed discussion of their influence will be given in ref. [4].

To illustrate the mapping procedure we consider the case of a single j orbit and of the operator $T^{(2)} = [a_j^+ \times a_j]^{(2)}$ defined by the fermion operators a_{jm}^+, a_{jm} (we omit the indices π and ν). For a single j orbit (or many j orbits with equal α_j's in eq. (1)), the n_d quantum number can be directly identified with the fermion seniority devided by 2, $n_d = v/2$. Then the ordinary reduction formulas can be written. For example

$$\langle j^n, v, J| \, |T^{(2)}| \, |j^n, v-2, J'\rangle$$
$$= \sqrt{\left(\frac{n}{2} - \frac{v}{2} + 1\right) \frac{2\Omega - n - v + 2}{2(\Omega - v + 1)}} \qquad (6)$$
$$\times \langle j^v, v, J| \, |T^{(2)}| \, |j^v, v-2, J'\rangle,$$

where $\Omega = j + \frac{1}{2}$. Since this operator changes the seniority by +2, it is mapped into a boson operator of the form $(d^+s)^{(2)}$. The first factor in eq. (6), $\sqrt{\frac{1}{2}n - \frac{1}{2}v + 1}$, is precisely equal to the matrix element of the s-boson annihilation operator s between the two states $|s^{1/2(n-v)+1}\rangle$ and $|s^{1/2(n-v)}\rangle$. Thus, it appears when calculating matrix elements of $(d^+s)^{(2)}$ in the s–d boson space. The second factor in eq. (6), however, is due to the Pauli principle and must be explicitly introduced into the corresponding boson operator. The operator $(d^+s)^{(2)}$ should be multiplied by

$$\sqrt{(2\Omega + 2 - n - v)/(2\Omega + 2 - 2v)}$$
$$= \sqrt{(2\Omega + 2 - 2n_s - 4n_d)/(2\Omega + 2 - 4n_d)}.$$

When the number n of protons (or neutrons) exceeds Ω, the numbers of s- and d-bosons are given by $2n_d = v$, $2(n_s + n_d) = 2\Omega - n$. The coefficient on the rhs of eq. (6) is, however, symmetric with respect to particles and holes. Thus, beyond the middle of the shell the factor $\sqrt{\frac{1}{2}(2\Omega - n) - \frac{1}{2}v + 1}$ is obtained from the boson matrix element while the first factor should be explicitly introduced. The latter one is still given by the same expression in the boson operators

$$\sqrt{(n - v + 2)/(2\Omega + 2 - 2v)}$$
$$= \sqrt{(2\Omega + 2 - 2n_s - 4n_d)/(2\Omega + 2 - 4n_d)}$$

as before. Looking now at the dependence on n and v of the seniority conserving matrix elements of $T^{(2)}$, we can write down the lowest order boson operator which corresponds to the complete fermion operator $T^{(2)}$ as follows:

$$Q^{(2)} = \frac{1}{\sqrt{5}} \langle j^2(D), J = 2| \, |T^{(2)}| \, |j^2(S), J = 0\rangle$$
$$\times \left\{ \sqrt{\frac{\Omega + 1 - N - n_d}{\Omega + 1 - 2n_d}} \, (d^+s)^{(2)} \right.$$
$$\left. + (s^+d)^{(2)} \sqrt{\frac{\Omega + 1 - N - n_d}{\Omega + 1 - 2n_d}} \right\} \qquad (7)$$
$$+ \frac{1}{\sqrt{5}} \langle j^2(D), J = 2| \, |T^{(2)}| \, |j^2(D), J = 2\rangle$$
$$\times \left(\frac{\Omega - 2N}{\Omega - 2n_d} \right) (d^+ \times d)^{(2)}.$$

Higher order operators can be constructed in a similar way. However, as it will be shown in detail in ref. [4], their matrix elements are usually small and thus can be, to a good approximation, neglected in practical calculations. The smallness of the higher order terms implies that the fermion matrix elements scale in a particular way for all states of a given v.

The boson proton–neutron interaction is given by the scalar product of $Q_\pi^{(2)}$ and $Q_\nu^{(2)}$. The matrix elements of the fermion operator $T^{(2)}$ which conserve seniority change sign at the middle of the proton (or neutron) shell due to the factor $(\Omega - n)/(\Omega - v)$. This change will not occur in the second term in eq. (7) since beyond the middle of the shell the boson number $N = n_s + n_d$ is determined by the number of *holes*. Therefore, this change of sign should be explicitly carried out in eq. (7) beyond the middle of the shell. On the other hand, the sign of the first term in eqs. (7) does not change. Therefore, for a single j-orbit, we expect to have symmetry between proton bosons and neutron bosons only if both protons and neutrons fill

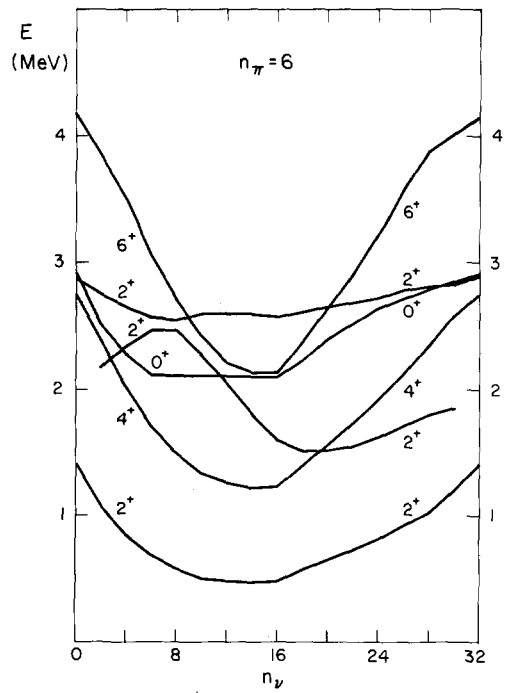

Fig. 1. Energy spectra of even–even nuclei, for fixed proton number, $n_\pi = 6$, and varying neutron number, $0 \leqslant n_\nu \leqslant 32$, in the single j-orbit approximation.

less than half their shells (particles) or both more than it (holes). When proton boson states are made of particles and neutron boson states of holes or vice versa, the hamiltonian does not even approximately commute with F-spin. In the general case (more than one j-orbit) breaking of F-spin is related to the relative sign and magnitude of the two terms in eq. (7) and it should be calculated in detail.

Once the boson hamiltonian has been constructed, it should be diagonalized in order to obtain spectra of even–even nuclei. An example of results of such a diagonalization is shown in fig. 1. The number of protons here was fixed, $n_\pi = 6$, and the neutron number was changed between $n_\nu = 0$ and $n_\nu = 32$. A single j orbit with $j = 31/2$ was assumed for both neutrons and protons. The boson hamiltonians H_π and H_ν were calculated for an attractive δ-potential (which is diagonal in the seniority scheme) and with strength adjusted as to produce a 0–2 separation in semi-magic nuclei of 1.4 MeV. The proton–neutron interaction was taken as $H_{\pi\nu} = -kT_\pi^{(2)} \cdot T_\nu^{(2)}$. This interaction gives rise to a proton boson–neutron boson interaction of the form $-\kappa\, Q_\pi^{(2)} \cdot Q_\nu^{(2)}$, where $Q_\pi^{(2)}$ and $Q_\nu^{(2)}$ have the form (7) with values of the coefficients of $(d^+s + s^+d)^{(2)}$ and $(d^+ \times d)^{(2)}$ appropriate to $j = 31/2$. In spite of these simplifications the results follow qualitatively the observed spectra in a wide variety of nuclei. In fig. 1 we can identify a region with $n_\nu \approx 4$ for which the spectrum is typical of an anharmonic vibrator (boson SU(5) picture [2]). Another region is obtained for $n_\nu \approx 10$–14 in which levels tend to the typical pattern of an axial rotor (boson SU(3) picture [2]). Finally for $n_\nu \approx 28$ the spectrum resembles that of a triaxial rotor (boson O(6) picture [2]). It seems remarkable that all these situations arise from approximate solutions of the shell model within a shell.

Finally we note that the agreement with experiment, which is only qualitative in fig. 1, can be made very quantitative if we renormalize the boson energies $\epsilon_{\pi(\nu)}$. The simple hamiltonian

$$H = \epsilon(n_{d_\pi} + n_{d_\nu}) - \kappa Q_\pi^{(2)} \cdot Q_\nu^{(2)}, \qquad (8)$$

with $\epsilon = 1.4$ MeV at $n_\nu = 0.32$; $\epsilon = 0.9$ MeV at $n_\nu = 2.30$; $\epsilon = 0.7$ MeV at $n_\nu = 4.28$; $\epsilon = 0.5$ MeV at $6 \leqslant n \leqslant 26$ and $\kappa = -0.2$ MeV throughout was used. The results are shown in fig. 2 together with the experimental data of the Ba isotopes. The renormalization of the single bo-

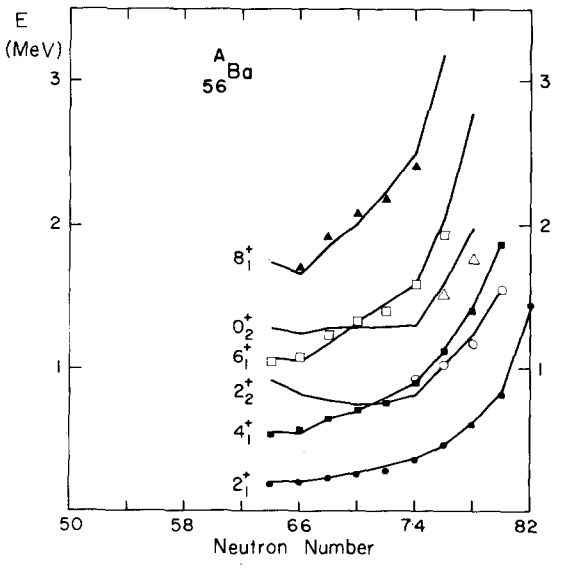

Fig. 2. Calculated energy spectra in barium isotopes using the hamiltonian (8). The dependence of the operators $Q_{\pi(\nu)}^{(2)}$ on particle number is given in eq. (7). The points, circles, squares and triangles are the experimental values.

son energies is probably due to the effect of the states outside the S–D subspace.

In conclusion, we have presented here an approximation scheme for large shell model calculations which allows microscopic calculations of collective states in heavy even–even nuclei. The correspondence with boson states makes these calculations much simpler and more transparent for understanding the physical picture of the resulting states. The scheme seems to describe the actual situation in many nuclei. In order to make it more quantitative, the simplified prescription (7) (which is valid in the case of a single j-orbit) should be replaced by a more detailed dependence on proton and neutron number. Such a dependence must be obtained from a more realistic study of single nucleon orbits which will determine, for each major shell, the actual values of the coefficients α_j and $\beta_{jj'}$ in eq. (1). As one can see from fig. 2 it seems that a simple renormalization of the boson energy goes already a long way in comparison with experiment. As more nuclei are investigated in this approach, the usefulness and scope of the model will be better determined.

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